Integrable non-commutative equations on quad-graphs. The consistency approach

A.I. Bobenko [*] Yu.B. Suris [†]

Abstract

We extend integrable systems on quad-graphs, such as the Hirota equation and the cross-ratio equation, to the non-commutative context, when the fields take values in an arbitrary associative algebra. We demonstrate that the three-dimensional consistency property remains valid in this case. We derive the non-commutative zero curvature representations for these systems, based on the latter property. Quantum systems with their quantum zero curvature representations are particular cases of the general non-commutative ones.

1 Introduction

The idea to use the $(d+1)$–dimensional consistency (or compatibility) of the discrete d–dimensional equations as the definition of their integrability was recently put forward in [3], [1] (and independently in [14]). This definition, apart of being conceptually transparent, has also other important theoretical advantages. So, finding the zero curvature representation for a given discrete system becomes an algorithmically solvable problem (recall that normally this was considered as a transcendental task whose successful solution is only possible with a large portion of luckyness in the guesswork). Also, in [1] it was demonstrated that the consistency criterium can be successfully used to classify integrable systems within certain ansätze.

In the present paper we give a further application of the consistency approach: we show that it works equally smoothly for *non-commutative* equations, where the participating fields live in an arbitrary associative (not necessary commutative) algebra $\mathcal{A}$ (over the field $\mathcal{K}$), and not just in $\mathbb{C}$, as in [3],

[*]Institut für Mathematik, Technische Universität Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany. E–Mail: bobenko@math.tu-berlin.de

[†]Institut für Mathematik, Technische Universität Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany. E–Mail: suris@sfb288.math.tu-berlin.de

[1]. We do not develop the corresponding classification, but rather consider several important examples, which generalize those already appeared in various applications, such as the quantum Hirota equation or the quaternionic cross–ratio equation. It turns out that finding the zero curvature representation in 2×2 matrices with entries from $\mathcal{A}$ does not hinge on the particular algebra $\mathcal{A}$ or on prescribing some particular commutation rules for fields in the neighboring vertices (like Weyl commutation relations in the traditional treatment of the quantum Hirota equation [8]). The fact that some commutation relations are preserved by the evolution, is thus conceptually separated from the integrability.

2 Basic setup

We start with a planar quad–graph $\mathcal{D}$, i.e. a cell decomposition of a surface, with all 2–cells being quadrilaterals. The sets of the vertices, edges, and faces of $\mathcal{D}$ (i.e. of its 0-, 1-, and 2-cells) will be denoted by $V(\mathcal{D})$, $E(\mathcal{D})$, and $F(\mathcal{D})$, respectively. The quad–graph $\mathcal{D}$ is supposed to carry a *labelling*, i.e. a function α on its edges which takes equal values on any two opposite edges of any elementary quadrilateral. The fields $x \in \mathcal{A}$ are assigned to vertices of $\mathcal{D}$. They take values in an arbitrary associative (in general non-commutative) algebra $\mathcal{A}$ with unit over the field $\mathcal{K}$.

Basic building blocks of systems on quad–graphs are equations on quadrilaterals of the type

$$Q(x, u, v, y; \alpha, \beta) = 0, \tag{1}$$

where $x, u, v, y \in \mathcal{A}$ are the fields assigned to the four vertices of the quadrilateral, and $\alpha, \beta \in \mathcal{K}$ are the parameters assigned to its edges, as shown on Fig. 1. We say that the equation (1) admits a *zero curvature representation* if

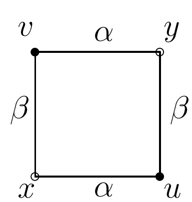

Figure 1: An elementary quadrilateral

to every oriented edge (x, u) carrying the label α there corresponds a matrix $L(u, x, \alpha, \lambda)$ depending on an arbitray (spectral) parameter $\lambda \in \mathcal{K}$ such that

$$L(x, u, \alpha, \lambda) = (L(u, x, \alpha, \lambda))^{-1}, \tag{2}$$

and for any elementary quadrilateral, as on Fig. 1, the equation (1) is equivalent to

$$L(x,v,\beta,\lambda)L(v,y,\alpha,\lambda)L(y,u,\beta,\lambda)L(u,x,\alpha,\lambda)=I, \tag{3}$$

that is,

$$L(y,u,\beta,\lambda)L(u,x,\alpha,\lambda)=L(y,v,\alpha,\lambda)L(v,x,\beta,\lambda). \tag{4}$$

3 Non-commutative Hirota equation

We start our considerations with the following equation:

$$yx^{-1}=f_{\alpha\beta}(uv^{-1}). \tag{5}$$

We require that this equation does not depend on how we regard the elementary quadrilateral. First of all, in general the elementary quadrilaterals are not supposed to be oriented in some consistent manner, which means that we cannot distinguish between left and right, so that (5) has to be equivalent to

$$yx^{-1}=f_{\beta\alpha}(vu^{-1}).$$

Therefore, we require that

$$f_{\alpha\beta}(A)=f_{\beta\alpha}(A^{-1}). \tag{6}$$

Second, the equation (5) should allow to exchange the roles of x and y, i.e. to be equivalent to

$$xy^{-1}=f_{\alpha\beta}(vu^{-1}).$$

Hence, we impose the following condition on the function $f_{\alpha\beta}$:

$$f_{\alpha\beta}(A^{-1})=(f_{\alpha\beta}(A))^{-1}. \tag{7}$$

Additionally, if one wants be able to exchange the roles of the pairs (x,y) and (u,v), then (5) should be equivalent to

$$uv^{-1}=f_{\beta\alpha}(xy^{-1}).$$

This leads to the following condition on the function $f_{\alpha\beta}$:

$$f_{\beta\alpha}(A)=f_{\alpha\beta}^{-1}(A^{-1}). \tag{8}$$

Here $f_{\alpha\beta}^{-1}$ stands for the inverse function to $f_{\alpha\beta}$, which has to be distinguished from the inversion in the algebra $\mathcal{A}$ in the formula (7).

All the conditions (6)–(8) are satisfied for the function which characterizes the *Hirota equation*:

$$f_{\alpha\beta}(A) = \frac{1-(\beta/\alpha)A}{(\beta/\alpha)-A}. \tag{9}$$

3.1 Three–dimensional consistency

Now we demonstrate that the non–commutative Hirota equation has a deep property of the three–dimensional consistency [3]. Consider an elementary cube of the three–dimensional lattice, as shown on Fig. 2.

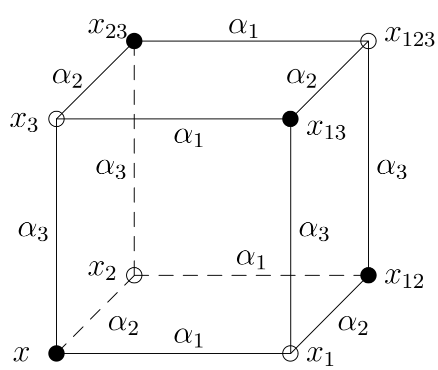

Figure 2: Elementary cube of the three–dimensional lattice

We assume that all edges of the elementary cube parallel to the axis number j ($j = 1, 2, 3$) carry the label α_j. Now, the fundamental *three–dimensional consistency* property should be understood as follows. Suppose that the values of the dependent variable are given at the vertex x and at its three neighbors x_1, x_2, and x_3. Then the Hirota equation (5) uniquely determines its values at x_{12}, x_{13}, and x_{23}. After that the Hirota equation delivers three *á priori* different values for the value of the dependent variable at the vertex x_{123}, coming from the faces $(x_1, x_{12}, x_{123}, x_{13})$, $(x_2, x_{23}, x_{123}, x_{12})$, and $(x_3, x_{13}, x_{123}, x_{23})$, respectively. The three–dimensional consistency means that *these three values for x_{123} actually coincide.*

Theorem 1. *The non–commutative Hirota equation is three–dimensionally consistent.*

First proof of Theorem 1. We give two proofs of this theorem. The first one is based on direct computations and is therefore more specific for the

Hirota equation. The second one paves the road to the derivation of the zero curvature representation from the equations governing the system, and is of a more general nature.

We have, by construction,

$$x_{ij}x^{-1} = f_{\alpha_i\alpha_j}(x_i x_j^{-1}),$$

and

$$x_{123}x_i^{-1} = f_{\alpha_j\alpha_k}(x_{ij}x_{ik}^{-1}), \tag{10}$$

where (i, j, k) is an arbitrary permutation of $(1, 2, 3)$. So, the three–dimensional consistency is equivalent to the equation

$$f_{\alpha_j\alpha_k}(x_{ij}x_{ik}^{-1})x_i = f_{\alpha_i\alpha_k}(x_{ij}x_{jk}^{-1})x_j,$$

or else to

$$f_{\alpha_j\alpha_k}\Big(f_{\alpha_i\alpha_j}(x_i x_j^{-1})(f_{\alpha_i\alpha_k}(x_i x_k^{-1}))^{-1}\Big) = \\ f_{\alpha_i\alpha_k}\Big(f_{\alpha_i\alpha_j}(x_i x_j^{-1})(f_{\alpha_j\alpha_k}(x_j x_k^{-1}))^{-1}\Big)x_j x_i^{-1}.$$

Taking into account that actually $f_{\alpha\beta}$ dependes only on β/α, we slightly abuse the notations and write $f_{\alpha\beta} = f_{\beta/\alpha}$. Denoting $\lambda = \alpha_j/\alpha_i$, $\mu = \alpha_k/\alpha_j$, and $A = x_i x_j^{-1}$, $B^{-1} = x_j x_k^{-1}$, and taking into account the property (7), we rewrite the above equation as

$$f_\mu\Big(f_\lambda(A)f_{\lambda\mu}(BA^{-1})\Big) = f_{\lambda\mu}\Big(f_\lambda(A)f_\mu(B)\Big)A^{-1}. \tag{11}$$

So, in order to prove the theorem, we have to demonstrate that the function (9) satisfies this functional equation for any $\lambda, \mu \in \mathcal{K}$ and for any $A, B \in \mathcal{A}$. In this proof we repeatedly use the identity

$$f_\lambda(CD^{-1}) = (D - \lambda C)(\lambda D - C)^{-1}.$$

The proof of the functional equation (11) is as follows.

$$\begin{aligned} & f_\mu\Big(f_\lambda(A)f_{\lambda\mu}(BA^{-1})\Big) \\ &= \Big((\lambda\mu A - B) - \mu f_\lambda(A)(A - \lambda\mu B)\Big)\Big(\mu(\lambda\mu A - B) - f_\lambda(A)(A - \lambda\mu B)\Big)^{-1} \\ &= \Big(\mu A(\lambda - f_\lambda(A)) - (1 - \lambda\mu^2 f_\lambda(A))B\Big)\Big(A(\lambda\mu^2 - f_\lambda(A)) - \mu(1 - \lambda f_\lambda(A))B\Big)^{-1} \end{aligned}$$

Next, we use the fact that

$$1 - \lambda f_\lambda(A) = A(\lambda - f_\lambda(A)).$$

This allows us to continue the chain of equations above:

$$\begin{aligned}
&= \Big(\mu(1-\lambda f_\lambda(A)) - (1-\lambda\mu^2 f_\lambda(A))B\Big)\Big(A(\lambda\mu^2 - f_\lambda(A)) - \mu A(\lambda - f_\lambda(A))B\Big)^{-1} \\
&= \Big(\mu - B - \lambda\mu f_\lambda(A)(1-\mu B)\Big)\Big(\lambda\mu(\mu - B) - f_\lambda(A)(1-\mu B)\Big)^{-1} A^{-1} \\
&= f_{\lambda\mu}\Big(f_\lambda(A) f_\mu(B)\Big)A^{-1}.
\end{aligned}$$

Theorem is proved. $\square$

Remark. It is difficult to write down an expression for x_{123} from which the symmetry with respect to permutations of indices $(1,2,3)$ would be apparent. For example, by simplifying (10) one can get:

$$\begin{aligned}
x_{123} &= \Big(\frac{\alpha_j}{\alpha_i} - x_i x_j^{-1}\Big)^{-1}\Big(\ell_{ij}x_i + \ell_{jk}x_k + \ell_{ki}x_i x_j^{-1} x_k\Big) \\
&\times \Big(\ell_{kj}x_i + \ell_{ik}x_j + \ell_{ji}x_k\Big)^{-1}\Big(\frac{\alpha_j}{\alpha_i} - x_i x_j^{-1}\Big)x_j\,, \qquad (12)
\end{aligned}$$

where

$$\ell_{ij} = \frac{\alpha_i}{\alpha_j} - \frac{\alpha_j}{\alpha_i}.$$

Of course, in the commutative case this expression becomes symmetric:

$$x_{123} = \frac{\ell_{ij}x_i x_j + \ell_{jk}x_j x_k + \ell_{ki}x_k x_i}{\ell_{kj}x_i + \ell_{ik}x_j + \ell_{ji}x_k},$$

but it is not so in the non–commutative case. On the other hand, one can rewrite (10) as

$$x_{ij}x_{ik}^{-1} = f_{\alpha_j\alpha_k}(x_{123}x_i^{-1}),$$

and a product of three such equations gives:

$$f_{\alpha_1\alpha_2}(x_{123}x_3^{-1}) f_{\alpha_3\alpha_1}(x_{123}x_2^{-1}) f_{\alpha_2\alpha_3}(x_{123}x_1^{-1}) = 1.$$

This equation for x_{123} is obviously symmetric with respect to permutations of indices. Moreover, it makes apparent that x_{123} depends on x_1, x_2, x_3 only, and not on x (this was called the "tetrahedron property" in [1]).

3.2 Zero curvature representation from three–dimensional consistency

Second proof of Theorem 1. We have to prove that the following three schemes for computing x_{123} lead to one and the same result:

- $(x, x_1, x_2) \mapsto x_{12}\,, \ \ (x, x_1, x_3) \mapsto x_{13}\,, \ \ (x_1, x_{12}, x_{13}) \mapsto x_{123}\,.$
- $(x, x_1, x_2) \mapsto x_{12}\,, \ \ (x, x_2, x_3) \mapsto x_{23}\,, \ \ (x_2, x_{12}, x_{23}) \mapsto x_{123}\,.$
- $(x, x_1, x_3) \mapsto x_{13}\,, \ \ (x, x_2, x_3) \mapsto x_{23}\,, \ \ (x_3, x_{13}, x_{23}) \mapsto x_{123}\,.$

We shall do this for the first two schemes only, since the rest is done similarly (or just by changing indices). The Hirota equation on the face (x, x_1, x_{13}, x_3),

$$x_{13}x^{-1} = f_{\alpha_3\alpha_1}(x_3 x_1^{-1}),$$

can be written as a formula which gives x_{13} as a fractional–linear transformation of x_3:

$$x_{13} = (\alpha_1 x_3 - \alpha_3 x_1)(\alpha_3 x_3 - \alpha_1 x_1)^{-1} x = L(x_1, x, \alpha_1, \alpha_3)[x_3], \qquad (13)$$

where

$$L(x_1, x, \alpha_1, \alpha_3) = \begin{pmatrix} \alpha_1 & -\alpha_3 x_1 \\ \alpha_3 x^{-1} & -\alpha_1 x^{-1} x_1 \end{pmatrix}. \qquad (14)$$

We use here the notation which is common for Möbius transformations on $\mathbb{C}$ represented as a linear action of the group $\mathrm{GL}(2,\mathbb{C})$. In the present case we define the action of the group $\mathrm{GL}(2,\mathcal{A})$ on $\mathcal{A}$ by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}[z] = (az+b)(cz+d)^{-1}, \qquad a,b,c,d,z \in \mathcal{A}.$$

It is easy to see that this is indeed the left action of the group, provided the multiplication in $\mathrm{GL}(2,\mathcal{A})$ is defined by the natural formula

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}.$$

Absolutely similarly to (13), we find:

$$x_{23} = L(x_2, x, \alpha_2, \alpha_3)[x_3]. \qquad (15)$$

From (15) we derive, by the shift in the direction of the first coordinate axis, the expression for x_{123} obtained by the first scheme above:

$$x_{123} = L(x_{12}, x_1, \alpha_2, \alpha_3)[x_{13}], \tag{16}$$

while from (13) we find the expression for x_{123} corresponding to the second scheme:

$$x_{123} = L(x_{12}, x_2, \alpha_1, \alpha_3)[x_{23}]. \tag{17}$$

Substituting (13), (15) on the right–hand sides of (16), (17), respectively, we represent the equality we want to demonstrate in the following form:

$$\begin{aligned} & L(x_{12}, x_1, \alpha_2, \alpha_3)L(x_1, x, \alpha_1, \alpha_3)[x_3] \\ = \;\; & L(x_{12}, x_2, \alpha_1, \alpha_3)L(x_2, x, \alpha_2, \alpha_3)[x_3]. \end{aligned} \tag{18}$$

We demonstrate that actually the stronger claim holds, namely that

$$L(x_{12}, x_1, \alpha_2, \alpha_3)L(x_1, x, \alpha_1, \alpha_3) = L(x_{12}, x_2, \alpha_1, \alpha_3)L(x_2, x, \alpha_2, \alpha_3). \tag{19}$$

Indeed, the 11 entries on both parts of this matrix identity are equal to $\alpha_1\alpha_2 - \alpha_3^2 x_{12}x^{-1}$. Equating 12 entries on both parts is equivalent to the Hirota equation of the face (x, x_1, x_{12}, x_2), and the same holds for the 21 entries. Finally, equating the 22 entries is equivalent to the condition that $x_{12}x^{-1}$ commutes with $x_2 x_1^{-1}$, and this is, of course, so in virtue of the Hirota equation. This finishes the second proof. □

Actually, Eq. (19) is nothing but the zero curvature representation of the non–commutative Hirota equation. It remains only to spell out the necessary construction which parallels the commutative one presented in [3].

To derive a zero–curvature representation for an equation on $\mathcal{D}$ of the type (1) possessing the property of the three–dimensional consistency, we extend the quad–graph $\mathcal{D}$ into the third dimension. This means that we consider the second copy $\mathcal{D}'$ of $\mathcal{D}$ and add edges connecting each vertex $x \in V(\mathcal{D})$ with its copy $x' \in V(\mathcal{D}')$. On this way we obtain a "three–dimensional quad–graph" $\mathbf{D}$, whose set of vertices is

$$V(\mathbf{D}) = V(\mathcal{D}) \cup V(\mathcal{D}'),$$

whose set of edges is

$$E(\mathbf{D}) = E(\mathcal{D}) \cup E(\mathcal{D}') \cup \{(x, x') : x \in V(\mathcal{D})\},$$

and whose set of faces is

$$F(\mathbf{D}) = F(\mathcal{D}) \cup F(\mathcal{D}') \cup \{(x, u, u', x') : x, u \in V(\mathcal{D})\}.$$

We extend the labelling to $E(\mathbf{D})$ in the following way: each edge $(x', u') \in E(\mathcal{D}')$ carries the same label as its counterpart $(x, u) \in E(\mathcal{D})$, while all "vertical" edges (x, x') carry one and the same label λ. This label plays the role of the spectral parameter.

Elementary building blocks of $\mathbf{D}$ are "cubes" $(x, u, y, v, x', u', y', v')$, as shown on Fig. 3. This figure is identical with Fig. 2, up to notations. Consider the equation (1) on the "vertical" face (x, u, u', x'):

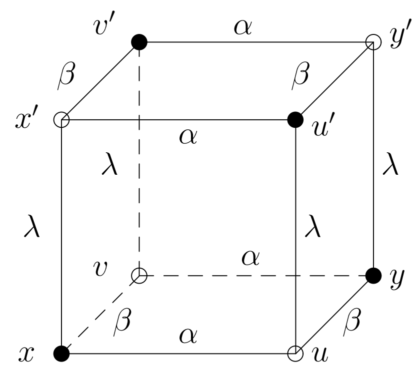

Figure 3: Elementary cube of the three–dimensional lattice

$$Q(x, u, x', u'; \alpha, \lambda) = 0,$$

and suppose that it gives u' as a fractional–linear transformation of x':

$$u' = L(u, x, \alpha, \lambda)[x'].$$

Then, due to the three–dimensional consistency, we have:

$$y' = L(y, u, \beta, \lambda)L(u, x, \alpha, \lambda)[x'] = L(y, v, \alpha, \lambda)L(v, x, \beta, \lambda)[x']\,.$$

This holds for arbitrary $x' \in \mathcal{A}$ and for all $\lambda \in \mathcal{K}$. If now some structural peculiarities of the matrices L allow us to conclude from the above that (4) holds, then the matrices L are the transition matrices of a zero curvature representation. (In the commutative case we could simply normalize the determinant of L in order to perform the last step.)

Theorem 2. *The Hirota equation admits a zero curvature representation with matrices from the loop group* $\mathrm{GL}(2, \mathcal{A})[\lambda]$*: the transition matrix along the (oriented) edge* (x, u) *carrying the label* α *is given by*

$$L(u, x, \alpha; \lambda) = \begin{pmatrix} \alpha & -\lambda u \\ \lambda x^{-1} & -\alpha x^{-1} u \end{pmatrix}. \tag{20}$$

Proof. Recall that the 11 entries of both matrix products in (4) are equal to $\alpha\beta - \lambda^2 y x^{-1}$. It is easy to see that if for $\mathcal{L}, \mathcal{M} \in \mathrm{GL}(2, \mathcal{A})$ there holds $\mathcal{L}[\xi] = \mathcal{M}[\xi]$ for all $\xi \in \mathcal{A}$, and $\mathcal{L}_{11} = \mathcal{M}_{11}$, then with necessity $\mathcal{L} = \mathcal{M}$. $\square$

3.3 Quantum Hirota equation

When speaking about solutions of equation like (1), one has in mind a suitably posed initial value problem for it. For a two–dimensional equation (1) Cauchy data (the values of the dependent variable x) should be prescribed along a one–dimensional path, i.e. on a sequence of points $\mathcal{C} = \{\mathfrak{z}_i\}_{i=i_0}^{i=i_1}$, where $i_0 \geq -\infty$, $i_1 \leq \infty$ and $\mathfrak{z}_i \in V(\mathcal{D})$. Whenever such a path contains three vertices of an elementary quadrilateral from $F(\mathcal{D})$, the equation (1) can be applied to get the value of x in the fourth vertex. This fourth vertex is then said to belong to $\mathcal{E}(\mathcal{C})$, the *evolution set of* $\mathcal{C}$. (Note that the original three vertices are not counted to $\mathcal{E}(\mathcal{C})$.) Continuing this process *ad infinitum*, we get a full set $\mathcal{E}(\mathcal{C})$ of vertices where the dependent variables are defined by a successive application of the equation to the initial data along $\mathcal{C}$. Of course, there are cases when the set $\mathcal{E}(\mathcal{C})$ is empty (think, for instance, of the case when $\mathcal{C}$ is the set of vertices of a regular quadratic lattice lying on a coordinate line); one is interested in $\mathcal{C}$ with a possibly large $\mathcal{E}(\mathcal{C})$. However, all data along $\mathcal{C}$ should be independent. This is formalized in the following definition: the path $\mathcal{C}$ is *space–like*, if $\mathcal{E}(\mathcal{C}) \cap \mathcal{C} = \emptyset$.

It is well–known that in the case of the regular square lattice the zigzag line as in Fig. 4 is a space–like path with $\mathcal{E}$ covering the whole lattice [4], [8]. Eq. (1) defines in this case the evolution in the vertical direction, i.e. the map $\{x_i\}_{i\in\mathbb{Z}} \mapsto \{\widetilde{x}_i\}_{i\in\mathbb{Z}}$. One often imposes the periodicity in the horizontal direction with an even period $2N$, in this case one is dealing with the regular square lattice on a cylinder rather than on the plane.

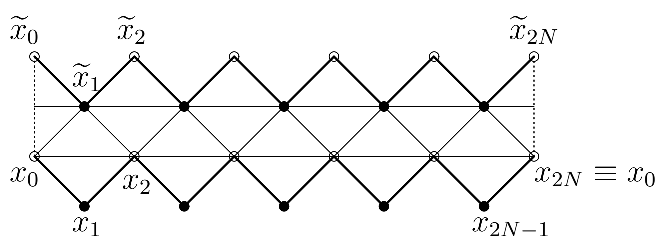

Figure 4: The Cauchy problem on a zigzag

A theory of the quantum Hirota equation on the regular square lattice was developed in [8]. As demonstrated there, in the non-commutative case

the map $\{x_i\}_{i\in\mathbb{Z}} \mapsto \{\widetilde{x}_i\}_{i\in\mathbb{Z}}$ preserves the following Weyl–like commutation relations:

$$\left\{\begin{array}{ll} [x_i, x_j] = 0, & j-i \ \text{ even}, \\ x_i x_j = q x_j x_i, & j-i>0 \ \text{ odd}. \end{array}\right. \tag{21}$$

Actually, this holds for an arbitrary space–like path in an arbitrary quad–graph $\mathcal{D}$, and for an arbitrary function $f_{\alpha\beta}$. So, this property has, in principle, nothing to do with integrability. The non–commutative Hirota equation with the Weyl commutation rules (21) along a space–like path is called the *quantum Hirota equation*.

We would like to stress that the quantum Hirota equation defines an evolution also in the multi–dimensional situation. This is due to its consistency property proven above for an arbitrary non–commutative Hirota system. So, in notations of Fig. 2, one can take as a Cauchy path the sequence $\{x, x_1, x_{12}, x_{123}\}$ with the commutation relations

$$[x, x_{12}] = [x_1, x_{123}] = 0, \quad x x_1 = q x_1 x, \quad x x_{123} = q x_{123} x, \quad x_{12} x_{123} = q x_{123} x_{12},$$

and get the values x_3 and x_{23} with the commutation relations

$$[x, x_{23}] = [x_3, x_{123}] = 0, \quad x x_3 = q x_3 x, \quad x x_{123} = q x_{123} x, \quad x_{23} x_{123} = q x_{123} x_{23},$$

in two different ways, according to two schemes:

- $(x_1, x_{12}, x_{123}) \mapsto x_{13}\,,\ (x, x_1, x_{13}) \mapsto x_3\,,\ (x_3, x_{13}, x_{123}) \mapsto x_{23}\,.$
- $(x, x_1, x_{12}) \mapsto x_2\,,\ (x_2, x_{12}, x_{123}) \mapsto x_{23}\,,\ (x, x_2, x_{23}) \mapsto x_3\,.$

Due to the three–dimensional consistency, these two schemes lead to identical results. This property separates the quantum Hirota equation among all other quantum equations of the type (5). As demonstrated above, this property allows us also to *derive* the quantum zero curvature representation found in [8].

It would be desirable to relate this property with another one [8], namely the Yang–Baxter relation

$$r(\lambda, u) r(\lambda\mu, v) r(\mu, u) = r(\mu, v) r(\lambda\mu, u) r(\lambda, v)$$

for the solution of the functional equation

$$\frac{r(\lambda, qw)}{r(\lambda, q^{-1}w)} = f_\lambda(w),$$

which also separates the Hirota equation among all other equations of the type (5) with $f_{\alpha\beta} = f_{\beta/\alpha}$.

4 Non-commutative cross–ratio equation

4.1 Equation and its different forms

Consider the system on a quad–graph $\mathcal{D}$, consisting of the following equations on elementary quadrilaterals:

$$(x-u)(u-y)^{-1}(y-v)(v-x)^{-1}=\frac{\alpha}{\beta}, \tag{22}$$

where $\alpha,\beta\in\mathcal{K}$. This equation was considered previously in two particular settings, when it has important geometrical applications: $\mathcal{A}=\mathbb{C}$, $\mathcal{K}=\mathbb{C}$ (discrete conformal maps; for the case of a regular square lattice and $\alpha/\beta=-1$ see, e.g., [15]; for the general case on arbitrary quad-graphs see [3]), and $\mathcal{A}=\mathbb{H}$, $\mathcal{K}=\mathbb{R}$ (discrete isothermic surfaces and their Darboux transformations, see [2], [11]). If $\mathcal{A}=\mathcal{C}(n)$, the Clifford algebra over $\mathcal{K}=\mathbb{R}$, this equation describes multi-dimensional isothermic nets, cf. [16].

In the notations with indices Eq. (22) can be put as

$$\alpha_1(x_{12}-x_1)(x_1-x)^{-1}=\alpha_2(x_{12}-x_2)(x_2-x)^{-1}. \tag{23}$$

This makes obvious the symmetry of this equation with respect to the simultaneous flip $x_1\leftrightarrow x_2$, $\alpha_1\leftrightarrow\alpha_2$, as well as to the simultaneous flip $x\leftrightarrow x_{12}$, $\alpha_1\leftrightarrow\alpha_2$. Several other forms of this equation and its consequences will be of interest for us. For instance, we transform (23) as

$$\alpha_1(x_{12}-x)(x_1-x)^{-1}-\alpha_1=\alpha_2(x_{12}-x)(x_2-x)^{-1}-\alpha_2$$

in order to arrive at the so–called three–leg form:

$$\alpha_1(x_1-x)^{-1}-\alpha_2(x_2-x)^{-1}=(\alpha_1-\alpha_2)(x_{12}-x)^{-1}. \tag{24}$$

Notice that by multiplying this equation by $x_{12}-x$ from the right, we eventually arrive at

$$\alpha_1(x_1-x)^{-1}(x_{12}-x_1)=\alpha_2(x_2-x)^{-1}(x_{12}-x_2), \tag{25}$$

which is thus demonstrated to be equivalent to (23), the fact non–obvious in the non–commutative context.

One could choose the point x_{12} as the common point of the three legs, and obtain instead of (24) the equation

$$\alpha_2(x_1-x_{12})^{-1}-\alpha_1(x_2-x_{12})^{-1}=(\alpha_2-\alpha_1)(x-x_{12})^{-1}.$$

Since the right–hand sides of this equation and of Eq. (24) coincide, we come to the following consequence of the basic equation:

$$\alpha_1(x_1-x)^{-1}-\alpha_2(x_2-x)^{-1}=\alpha_1(x_{12}-x_2)^{-1}-\alpha_2(x_{12}-x_1)^{-1}. \tag{26}$$

4.2 Three–dimensional consistency

Theorem 3. *The non–commutative cross–ratio equation is three–dimensionally consistent.*

Proof. We proceed as in the second proof of Theorem 1. From the cross–ratio equation of the face (x, x_1, x_{13}, x_3),

$$\alpha_1(x_{13} - x_1)(x_1 - x)^{-1} = \alpha_3(x_{13} - x_3)(x_3 - x)^{-1}$$

we derive:

$$\alpha_1\alpha_3^{-1}(x_{13} - x_1)(x_1 - x)^{-1}(x_3 - x) = (x_{13} - x_1) + (x_1 - x_3),$$

which is equivalent to

$$(x_{13} - x_1)\Big(1 + \alpha_1\alpha_3^{-1}(x - x_1)^{-1}(x_3 - x)\Big) = x_3 - x_1 = (x_3 - x) + (x - x_1).$$

This can be put in the matrix form as

$$x_{13} - x_1 = L(x_1, x, \alpha_1, \alpha_3)[x_3 - x], \tag{27}$$

where

$$L(x_1, x, \alpha_1, \alpha_3) = \begin{pmatrix} 1 & x - x_1 \\ \alpha_1\alpha_3^{-1}(x - x_1)^{-1} & 1 \end{pmatrix}. \tag{28}$$

Similarly,

$$x_{23} - x_2 = L(x_2, x, \alpha_2, \alpha_3)[x_3 - x], \tag{29}$$

From (29), (27) we derive, by the shift in the direction of the first, resp. the second coordinate axis, the expressions for x_{123} obtained by the first, resp. the second scheme above:

$$\begin{aligned} x_{123} - x_{12} &= L(x_{12}, x_1, \alpha_2, \alpha_3)[x_{13} - x_1], & (30)\\ x_{123} - x_{12} &= L(x_{12}, x_2, \alpha_1, \alpha_3)[x_{23} - x_2]. & (31) \end{aligned}$$

Substituting (27), (29) on the right–hand sides of (30), (31), respectively, we represent the equality we want to demonstrate in the following form:

$$\begin{aligned} & L(x_{12}, x_1, \alpha_2, \alpha_3)L(x_1, x, \alpha_1, \alpha_3)[x_3 - x] \\ = \; & L(x_{12}, x_2, \alpha_1, \alpha_3)L(x_2, x, \alpha_2, \alpha_3)[x_3 - x]. \end{aligned} \tag{32}$$

This is a consequence of a stronger claim:

$$L(x_{12},x_1,\alpha_2,\alpha_3)L(x_1,x,\alpha_1,\alpha_3)=L(x_{12},x_2,\alpha_1,\alpha_3)L(x_2,x,\alpha_2,\alpha_3). \quad (33)$$

Indeed, the 12 entries on the both sides are equal to $x-x_{12}$. Equating the 11 entries is equivalent to Eq. (23), equating the 22 entries is equivalent to the (inverted) Eq. (25), and equating the 21 entries is equivalent to Eq. (26). This finishes the proof. □

Remark. As in the case of the Hirota equation, this proof does not lead to an expression for x_{123} which would make the claim self–evident. However, also in this case the three–leg form of equations comes to help. Namely, summing up the equations

$$\alpha_i(x_{123}-x_{jk})^{-1}-\alpha_j(x_{123}-x_{ik})^{-1}=(\alpha_i-\alpha_j)(x_{123}-x_k)^{-1},$$

we come to the equation

$$(\alpha_2-\alpha_3)(x_{123}-x_1)^{-1}+(\alpha_3-\alpha_1)(x_{123}-x_2)^{-1}+(\alpha_1-\alpha_2)(x_{123}-x_3)^{-1}=0, \quad (34)$$

which makes two things obvious: first, that x_{123} depends only on x_1, x_2, x_3 and not on x (tetrahedron property), and second, the symmetry of the resulting x_{123} with respect to permutations of indices $(1,2,3)$. Comparing (34) with (24), we see that the former equation is again of the cross–ratio type: it is equivalent, e.g., to

$$(x_{123}-x_1)(x_1-x_3)^{-1}(x_3-x_2)(x_2-x_{123})^{-1}=(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)^{-1}. \quad (35)$$

4.3 Zero curvature representation

Setting $\alpha_3=\lambda^{-1}$ in the proof of Theorem 3, we come to the following statement.

Theorem 4. *The cross–ratio equation admits a zero curvature representation with matrices from the loop group* $\mathrm{GL}(2,\mathcal{A})[\lambda]$*: the transition matrix along the (oriented) edge* (x,u) *carrying the label* α *is given by*

$$L(u,x,\alpha;\lambda)=\begin{pmatrix} 1 & x-u \\ \lambda\alpha(x-u)^{-1} & 1 \end{pmatrix}. \quad (36)$$

An essential point is, we stress it again, that this zero curvature representation was *derived* from the equations of the system, without any additional information. Moreover, the proof of Theorem 3 shows again that the the zero curvature representation not only follows from the three–dimensional consistency, but is, in turn, instrumental in establishing it.

5 Concluding remarks

The present paper has to be considered in the context of the ongoing study of non-commutative integrable systems [5], [6], [13] which puts quantum integrable systems on a more general basis (see also [12], [9]). We expect that discrete integrable non-commutative systems of the sort considered in this paper are of a fundamental importance, just like it is the case in the commutative context [3]. It will be important to extend the classification results of [1] to the non-commutative case, and to get complete lists of discrete integrable equations. Also, a more thorough understanding of the quantum case and the origin of its specific (Yang–Baxter) structures is desirable. This could lead also to a new approach to classification of solutions of the Yang–Baxter equation.

Acknowledgment. This research was partly supported by DFG (Deutsche Forschungsgemeinschaft) in the frame of SFB 288 "Differential Geometry and Quantum Physics".

References

[1] V.E.Adler, A.I.Bobenko, Yu.B.Suris. Classification of integrable equations on quad–graphs. The consistency approach. *Preprint*, `http://www.arXiv.org/abs/nlin.SI/0202024`.

[2] A.I.Bobenko, U.Pinkall. Discrete isothermic surfaces. *J. Reine Angew. Math*, 1996, **475**, 187–208.

[3] A.I.Bobenko, Yu.B.Suris. Integrable systems on quad-graphs. *Int. Math. Res. Notices*, 2002, No 11, 573–611.

[4] H.W.Capel, F.W.Nijhoff, V.G.Papageorgiou. Complete integrability of Lagrangian mappings and lattices of KdV type. *Phys. Lett. A*, 1991, **155**, 377–387.

[5] P.Etingof, I.Gelfand, V.Retakh. Factorization of differential operators, quasideterminants, and nonabelian Toda field equations. *Math. Res. Lett.*, 1997, **4**, 413–425.

[6] P.Etingof, I.Gelfand, V.Retakh. Nonabelian integrable systems, quasideterminants, and Marchenko lemma. *Math. Res. Lett.*, 1998, **5**, 1–12.

[7] L.D.Faddeev, R.M.Kashaev, A.Yu.Volkov. Strongly coupled quantum discrete Liouville theory. I: Algebraic approach and duality. *Commun. Math. Phys.*, 2001, **219**, 199–219.

[8] L.D.Faddeev, A.Yu.Volkov. Hirota equation as an example of an integrable symplectic map. *Lett. Math. Phys.*, 1994, **32**, 125–135.

[9] I.Gelfand, V.Retakh. Quasideterminants. I. *Selecta Math.*, 1997, **3**, 517–546.

[10] R.Hirota. Nonlinear partial difference equations. I. A difference analog of the Korteweg–de Vries equation. III. Discrete sine-Gordon equation. *J. Phys. Soc. Japan*, 1977, **43**, 1423–1433, 2079–2086.

[11] U.Hertrich-Jeromin, T.Hoffmann, U.Pinkall. A discrete version of the Darboux transform for isothermic surfaces. – In: *Discrete Integrable Geometry and Physics*, Ed. A.I.Bobenko, R. Seiler, Oxford: Clarendon Press, 1999, 59–79.

[12] D.Krob, B.Leclerc. Minor identities for quasi-determinants and quantum determinants. *Comm. Math. Phys.*, 1995, **169**, 1–23.

[13] B.A.Kupershmidt. KP or mKP. Noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems. Providence, RI: AMS, 2000.

[14] F.W.Nijhoff. Lax pair for the Adler (lattice Krichever-Novikov) system. *Phys. Lett. A*, 2002, **297**, 49–58.

[15] F.W.Nijhoff, H.W.Capel. The discrete Korteweg-de Vries equation. *Acta Appl. Math.*, 1995, **39**, 133–158.

[16] W.K.Schief. Isothermic surfaces in spaces of arbitrary dimension: integrability, discretization, and Bäcklund transformations – a discrete Calapso equation. *Stud. Appl. Math.*, 2001, **106**, 85–137.